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THE DOMINATION INEQUALITY CHAIN FOR INDEPENDENCE SYSTEMS

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-Introduction

In classical Graph Theory the concepts of Independence and Domination are two of the best studied concepts. For a graph $G = (V, E)$ we say that a set of vertices $S \subseteq V$ is independent if no two vertices in S are adjacent. This concept has many useful applications if we want to place objects and not have interference between them. We say the set of vertices S dominates a vertex $v \in V$ if v is in S ,or v is adjacent to a vertex in S, and we say S is a dominating set if S dominates every vertex in the graph G .Domination has many useful applications when looking at covering problems, where you want every location to be connected to something in the set. In 1978, Cockayne et al. defined an inequality chain of parameters based on independence, domination and irredundance. This chain describes how the ranges of maximal independent and irredundant sets, interacts with the range of minimal dominating sets. In the classical form it is written as:

Fig.1x

$$ir(G) \leq \gamma(G) \leq i(G) \leq \beta(G) \leq \Gamma(G) \leq IR(G)$$

Where, $ir(G)$, and $IR(G)$ represents the minimum and maximum size of maximal irredundant sets, $\gamma(G)$, and $\Gamma(G)$ represent the minimum and maximum size of minimal dominating sets, and $i(G)$ and $\beta(G)$ represent the minimum and maximum size of maximal independent sets.

Since the original inequality chain was presented, many other graph parameters have been added to the chain, as well as many special cases of how these values interact with each other. In recent years a similar inequality chain has been produced for other concepts of independence and domination. In particular for Cycles, Odd cycles, and Maximal Cliques. This white paper presents a high level discussion of how these results can be extended to an broad range of properties, what that means in terms of potential applications, and some interesting special cases.

-Background and Definitions

We begin with a discussion of an abstract set system, independence systems. An independence system $S = (E,I)$ is composed of a finite set E, called the ground set, and a set of subsets of E, denoted by I, called the independent sets, where I is not empty, and has the property that it is hierarchical. That is, if $X \in I$ then every subset of X is also in I. One important property to note of independence systems is that if $X \notin I$, then every subset of E that contains X as a subset is also not in I. Together this gives rise to the concepts of bases, and circuits. A basis of an independence system S is a maximal set in I, that is, it is in I and no set which contains it is also in I. Similarly, a circuit is a minimal set not in I, that is, it is a set not in I where every subset is in I.

Since I has the hierarchical property this implies that the independence system $S = (E,I)$, can be defined by the set of all bases denoted as $B(S)$, or equivalently by the set of all circuits denoted as $C(S)$. That is, the independence system can be defined by the largest elements in I, or by the smallest subsets of E that are not in I.

We extend the definitions of the domination inequality chain to similar properties for independence systems, and justify this extension by showing how the definition applies to classical independence and domination. In particular for an independence system $S = (E,I)$, with bases $B(I)$, and circuits $C(I)$ we say a set $V \subseteq E$ is independent if $V \in I$. We denote the following parameters by:

Fig.2x

$$i(S) = min\{|V| : V \in B(S)\},$$
$$\beta(G) = max\{|V| : V \in I\} = max\{|V| : V \in B(S)\}$$

We say $V \subseteq E$ dominates some $s \in S$ if $s \in V$, or there is a circuit $U \in C(I)$, where $s \in U$, and $U \subseteq V \bigcup \{s\}$. That is if s is not in V then, the set $V \bigcup \{s\}$ contains a circuit that V, without s, does not contain. We say that a set V is a dominating set if V dominates every vertex in E and we denote the set of all minimal dominating sets of S by $D(S)$. Note that if a set V dominates some $s \in E$, then every set which contains V also dominates s, this implies that every dominating set can be obtained from the minimal dominating sets. We denote the following parameters by:

Fig.3x

$$\gamma(S) = \min\{|V| : V \in D(S)\},$$
$$\Gamma(S) = \max\{|V| : V \in D(S)\}.$$

Let $v \in V$, we say that v is irredundant in V if there is some $u \in E$ which is dominated by V but is not dominated by $V \setminus \{v\}$. Note that u could be equal to v. We say V is irredundant if every element of V is irredundant. Note that, trivially, the empty set is irredundant. Denote the set of all maximal irredundant sets by $F(S)$. We denote the following parameters by:

Fig.4x

$$ir(S) = \min\{|V| : V \in F(S)\},$$
$$IR(S) = \max\{|V| : V \in F(S)\}.$$

In order to justify, and explain, these new definitions they need to be an extension of the original definitions. In particular, given a graph $G=(V,E)$ we note that the circuits are now exactly the edges of the graph. Thus, the classical definitions of independence, domination and irredundance becomes special cases of the new definition when the circuits are the edges of a graph.

-Domination Inequality Chain

We now show that the generalizations of independence, domination and irredundance produces the same inequality chain. We begin with a few basic observations. Given an independence system $S=(E,I)$:

Fig.5x

$$i(S) \le \beta(S),$$
$$\gamma(S) \le \Gamma(G),$$
$$ir(S) \le IR(S).$$

These three inequalities follow directly from the definitions, where the lower bound is defined by a minimum over a set, and the upper bound is defined by the maximum over the same set. To build the other relations we follow the original proof by showing that every basis is a dominating set, and every minimal dominating set is an irredundant set.

Let $V \subseteq E$ be a maximal independent set, that is a basis. To show that it is a dominating set it suffices to show that it dominates every $u \in E$. Note that if $u \in V$ it is dominated trivially, thus we will consider when $u \notin V$. Since V is a maximal set, V

$\bigcup \{u\}$ can not be independent, and thus contains a circuit, which, since the circuit was not contained in V , must also contain u . Thus, the set V dominates u , and since u was arbitrary V dominates every $u \notin V$ and hence is a dominating set. To get minimality, note that for any $v \in V$ the set $V \setminus \{v\}$ does not dominate v as V does not contain a circuit. Thus, V is not only dominating, but the removal of any element stops it from being a dominating set, and thus it is a minimal dominating set. Thus, for every basis, the its size is bounded above by $\Gamma(S)$ and is bounded below by $\gamma(S)$ and thus we have:

Fig.6x

$$\gamma(S) \leq i(S) \leq \beta(S) \leq \Gamma(S)$$

Now, let $V \subseteq E$ be a minimal dominating set. To show that it is an irredundant set we let $v \in V$. If $V \setminus \{v\}$ is still a dominating set then V was not a minimal dominating set. Thus, $V \setminus \{v\}$ does not dominate some element and thus, v is irredundant in V . To obtain maximality, for some $u \notin V$ since V is dominating $V \bigcup \{u\}$ is also dominating, and hence does not dominate any elements that were not dominated by V and hence u is not irredundant in $V \bigcup \{u\}$ and thus V is a maximal irredundant set. Since every minimal dominating set is a maximal irredundant set we get the following inequalities:

Fig.7x

$$ir(S) \leq \gamma(S) \leq \Gamma(S) \leq IR(S)$$

Together with the earlier inequality we get the desired inequality chain:

Fig.8x

$$ir(S) \leq \gamma(S) \leq i(S) \leq \beta(S) \leq \Gamma(S) \leq IR(S)$$

-Applications

While these concepts seem to be highly abstract, to the point of uselessness perhaps, the real world applications of independence and domination have been studied. However, in order to apply results to similar concepts, often what is required is the restating of the problem into the original graph language. In this new, more general, setting any property that is hierarchical defines an independence system and thus has associated logical structures, and tools, associated with it and the resulting concepts of domination and irredundance. Similarly, any time we wish to obtain a maximal set of objects, with restrictions, as long as the restrictions can be phrased as a set of circuits there is an associated independence system.

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